** Methods of integration:

(1) Integration by substitution.
(2) Transformation of trigonometric functions.
(3) Powers of trigonometric functions.
(4) Even powers of sines and cosines.
(5) Integration by parts.
(6) Successive integration by parts.
(7) Some trigonometrical and hyperbolic substitutions.
(8) Completing the squares.
(9) Miscellanous substitutions.
(10) Integration using partial fractions.

هنمسك كل طريقة من دول ونشوف المسائل المختلفة عليها

by
Eng. Mohammed Emad

(1) Integration by substitution:

It's a technique that improve our use of the fundamental integration formulas. This technique of substitution involves a change of variable which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Evaluate:

(i) $I = \int [f(x)]^n \cdot f'(x)\, dx \longrightarrow \frac{[f(x)]^{n+1}}{n+1}$

Ans:

Put $f(x) = u \Rightarrow f'(x)\cdot dx = du$,

Then,

$$I = \int u^n \cdot du = \frac{u^{n+1}}{n+1} + C$$

$$= \frac{[f(x)]^{n+1}}{n+1} + C \quad , n \neq -1$$

(ii) $J = \int \frac{f'(x)}{f(x)}\, dx \longrightarrow = \ln f(x)$

Ans:

Put $u = f(x) \Rightarrow du = f'(x)\, dx$

$$\therefore J = \int \frac{du}{u} = \ln|u| + C$$

$$= \boxed{\ln|f(x)| + C}$$

From Pervious example, we can find the following integrals:

1. $\int \sin^n x \cdot \cos x \, dx$.

Ans Put $u = \sin x \Rightarrow du = \cos x \cdot dx$, Then

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$$\int \sin^n x \cdot \cos x \, dx = \int u^n \cdot du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{(\sin x)^{n+1}}{n+1} + C} \quad , n \neq -1$$

2. $\int \cos^n x \sin x \, dx$.

Ans: Put $u = \cos x \Rightarrow du = -\sin x \, dx$, Then

$$\int \cos^n x \cdot \sin x \, dx = -\int u^n \cdot du = -\frac{u^{n+1}}{n+1} + C = \boxed{-\frac{(\cos x)^{n+1}}{n+1} + C}$$

$n \neq -1$

3. $\int \tan^n x \cdot \sec^2 x \, dx$

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Ans: Put $u = \tan x \Rightarrow du = \sec^2 x \cdot dx$, Then

$$\int \tan^n x \cdot \sec^2 x \, dx = \int u^n \cdot du = \frac{u^{n+1}}{n+1} = \boxed{\frac{(\tan x)^{n+1}}{n+1} + C} \quad , n \neq -1$$

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4. $\int \sec^{n+1} x \cdot \tan x \, dx$

Ans: Put $u = \sec x \Rightarrow du = \sec x \cdot \tan x \, dx$

$$\therefore \int (\sec x)^n \cdot \sec x \cdot \tan x \, dx = \int u^n \cdot du = \frac{u^{n+1}}{n+1} + C$$

$$= \boxed{\frac{(\sec x)^{n+1}}{n+1} + C} \quad , n \neq -1$$

5. $\int \sinh^n x \cdot \cosh x \, dx$.

Ans: Put $u = \sinh x \Rightarrow du = \cosh x \, dx$, Then

$$\int \sinh^n x \cdot \cosh x \, dx = \int u^n \cdot du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{(\sinh x)^{n+1}}{n+1} + C} \quad , n \neq -1$$

6. $\int \cosh^n x \cdot \sinh x \, dx$

Ans: Put $u = \cosh x \Rightarrow du = \sinh x \, dx$, Then

$$\int \cosh^n x . \sinh x \, dx = \int u^n . du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{(\cosh x)^{n+1}}{n+1} + C}, \quad n \neq -1$$

7. $\int \tanh^n x \cdot \text{sech}^2 x \, dx$.

Ans: Put $u = \tanh x \Rightarrow du = \text{sech}^2 x \, dx$, Then

$$\int \tanh^n x . \text{sech}^2 x \, dx = \int u^n . du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{(\tanh x)^{n+1}}{n+1} + C}, \quad n \neq -1$$

8. $\int \text{sech}^{n+1} x \cdot \tanh x \, dx$

Ans: Put $u = \text{sech}\, x \Rightarrow du = -\text{sech}\, x . \tanh x . dx$, Then

$$\int \text{sech}^{n+1} x \cdot \tanh x \, dx = -\int u^n . du = -\frac{u^{n+1}}{n+1} + C = \boxed{-\frac{\text{sech}^{n+1} x}{n+1} + C}, \quad n \neq -1$$

9. $\int \tan x \, dx$

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Ans: let $I = \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx$,

Put $u = \cos x \Rightarrow du = -\sin x \, dx$. Then

$$I = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C = \boxed{\ln|\sec x| + C}$$

10. $\int \cot x \, dx$

Ans: let $I = \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$,

Put $u = \sin x \Rightarrow du = \cos x \, dx \Rightarrow I = \int \frac{du}{u} = \ln|u| + C$

$$\therefore \boxed{I = \ln|\sin x| + C}$$

11. $\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} dx = \boxed{\ln|\cosh x| + C}$

12. $\int \coth x \, dx = \int \frac{\cosh x}{\sinh x} dx = \boxed{\ln|\sinh x| + C}$

Example: If $\int f(u)\,du = F(u) + C$, Find $\int f(ax+b)\,dx$

Ans: Put $u = ax+b \Rightarrow du = a\,dx \Rightarrow dx = \frac{du}{a}$

$$\therefore \int f(ax+b)\,dx = \int f(u)\cdot\frac{du}{a} = \frac{1}{a}F(u) + C = \boxed{\frac{1}{a}F(ax+b) + C}$$

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Find:

(1) $\int (ax+b)^n\,dx$

Ans: Put $u = ax+b \Rightarrow du = a.dx \Rightarrow dx = \frac{1}{a}.du$, Then

$$\int (ax+b)^n\,dx = \int u^n\cdot\frac{1}{a}\cdot du = \frac{1}{a}\cdot\frac{u^{n+1}}{n+1} = \boxed{\frac{(ax+b)^{n+1}}{a(n+1)}}\ ,\ n \neq -1$$

(2) $\int \frac{dx}{ax+b}$

Ans: Put $u = ax+b \Rightarrow du = a.dx \Rightarrow dx = \frac{1}{a}du$, Then

$$\int \frac{dx}{ax+b} = \frac{1}{a}\int\frac{du}{u} = \frac{1}{a}\ln|u| + C = \boxed{\frac{1}{a}\ln|ax+b| + C}$$

(3) $\int (x^2+b)^n\,x\,dx$

Ans: Put $u = x^2+b \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x.dx$, Then

$$\int (x^2+b)^n\,x\,dx = \frac{1}{2}\int u^n\,du = \frac{1}{2}\,\frac{u^{n+1}}{n+1} + C = \boxed{\frac{(x^2+b)^{n+1}}{2(n+1)} + C}\ ,\ n \neq -1$$

(4) $\int \frac{x}{x^2+b}\,dx$

Ans: Put $u = x^2+b \Rightarrow du = 2x\,dx \Rightarrow \frac{1}{2}du = x.dx$, Then

$$\int \frac{x}{x^2+b}\,dx = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|x^2+b| + C}$$

(5) $\int \frac{e^x}{1+e^{2x}}\,dx$

Ans: Put $u = e^x \Rightarrow du = e^x\,dx$. Then

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int\frac{du}{1+u^2} = \tan^{-1}u + C = \boxed{\tan^{-1}(e^x) + C}$$

(6) $\int x.\sqrt{1+x}\, dx$.

Ans: Put $u = 1+x \Rightarrow x = u-1 \Rightarrow dx = du$, Then

$$\int x\sqrt{1+x}\, dx = \int (u-1).u^{\frac{1}{2}}\, du = \int u^{\frac{3}{2}} du - \int u^{\frac{1}{2}} du$$

$$= \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} + C$$

$$= \boxed{\frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + C}$$

(7) $\int \frac{x\, dx}{x^2+a^2}$.

Ans: Put $u = x^2+a^2 \Rightarrow du = 2x\, dx$, Then

$$\int \frac{x\, dx}{x^2+a^2} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln[x^2+a^2] + C}$$

(8) $I = \int \frac{x^2+2x}{x^3+3x^2+4} dx$.

Ans: Put $u = x^3+3x^2+4 \Rightarrow du = (3x^2+6x)\, dx$, or

$du = 3(x^2+6x)\, dx$, Then

$$I = \frac{1}{3}\int \frac{du}{u} \Rightarrow I = \frac{1}{3}\ln|u| + C = \boxed{\frac{1}{3}\ln|x^3+3x^2+4| + C}$$

(9) $I = \int \frac{\ln x}{x} dx$.

Ans: Put $u = \ln x \Rightarrow du = \frac{dx}{x}$, Then

$$I = \int u\, du = \frac{1}{2}u^2 + C = \boxed{\frac{1}{2}(\ln x)^2 + C}$$

(10) $I = \int \frac{(\ln x)^n}{x} dx$.

Ans: Put $u = \ln x \Rightarrow du = \frac{dx}{x}$, Then

$$I = \int u^n\, du = \frac{u^{n+1}}{n+1} + C = \boxed{\frac{[\ln x]^{n+1}}{n+1} + C}$$

(11) $K = \int \frac{dx}{x \ln x}$

Ans: Put $u = \ln x \Rightarrow du = \frac{dx}{x}$, Then

$$K = \int \frac{du}{u} = \ln|u| + C = \boxed{\ln|\ln x| + C}$$

Evaluate:

(ii) $I = \int \sec x \, dx$

Ans:

Multiply numerator and denominator by $(\sec x + \tan x)$, Then

$$I = \int \frac{\sec^2 x + \sec x . \tan x}{\sec x + \tan x} \, dx .$$

Put $u = \sec x + \tan x \Rightarrow du = (\sec x . \tan x + \sec^2 x) \, dx$, Then

$$I = \int \frac{du}{u} = \ln|u| + C \Rightarrow I = \ln|\sec x + \tan x| + C$$

$$\therefore \boxed{\int \sec x \, dx = \ln|\sec x + \tan x| + C} \rightarrow$$

يمكن استخدامها مباشرة في المسائل.

Another sol.n: حل أطول شوية

$$I = \int \sec x \, dx = \int \frac{1}{\cos x} \, dx = \int \frac{\cos x}{\cos^2 x} \, dx = \int \frac{\cos x}{1 - \sin^2 x} \, dx$$

Put $u = \sin x \Rightarrow du = \cos x \, dx$, Then

$$I = \int \frac{du}{1 - u^2} \Rightarrow \text{using Partial fraction, Then}$$

$$\frac{1}{1 - u^2} = \frac{1}{(1-u)(1+u)} = \frac{A}{(1-u)} + \frac{B}{(1+u)} ,$$

$, \ A(1+u) + B(1-u) = 1$

Put: $u = -1 \Rightarrow 2B = 1 \Rightarrow B = 0.5$

$u = 1 \Rightarrow 2A = 1 \Rightarrow A = 0.5$

$$\therefore I = 0.5 \int \frac{du}{1+u} + 0.5 \int \frac{du}{1-u} = 0.5 \left[\ln|1+u| - \ln|1-u| \right]$$

$$= 0.5 \ln\left|\frac{1+u}{1-u}\right| + C = 0.5 \ln\left|\frac{1+\sin x}{1-\sin x}\right| + C \longrightarrow (1) ,$$

$$\frac{1+\sin x}{1-\sin x} * \frac{1+\sin x}{1+\sin x} = \frac{(1+\sin x)^2}{1-\sin^2 x} = \frac{(1+\sin x)^2}{\cos^2 x}$$

$$= \left(\frac{1+\sin x}{\cos x}\right)^2 = (\sec x + \tan x)^2 \text{ sub. in } (1)$$

$$\therefore \boxed{I = \ln|\sec x + \tan x|} + C$$

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(ii) $J = \int \csc x \, dx$

Ans: Multiply numerator & denominator by $(\csc x - \cot x)$, Then

$$J = \int \frac{\csc^2 x - \csc x \cot x}{\csc x - \cot x} \, dx$$

Put $u = \csc x - \cot x \Rightarrow du = (-\csc x \cdot \cot x + \csc^2 x)\, dx$, Then

$J = \int \frac{du}{u} = \ln|u| + C$, Then

$$\boxed{\int \csc x \, dx = \ln|\csc x - \cot x| + C}$$

(iii) $I = \int \frac{\tan(\ln x)}{x} \, dx$

Ans: Put $u = \ln x \Rightarrow du = \frac{dx}{x}$, Then

$I = \int \tan(u) \, du = \ln|\sec u| + C$

$= \boxed{\ln|\sec(\ln x)| + C}$

(iv) $J = \int \frac{\cos 2x}{\sin^2 2x} \, dx$

Ans: Put $u = \sin 2x \Rightarrow du = 2 \cos 2x \, dx$

$\therefore J = \frac{1}{2} \int \frac{du}{u^2} = \frac{1}{2} \int u^{-2} \, du = \frac{1}{2} \frac{u^{-1}}{-1} + C = \frac{-1}{2u} = \boxed{\frac{-1}{2 \sin 2x} + C}$

(v) $I = \int \frac{4}{x^2 + a^2} \, dx$

Ans: $\boxed{I = \frac{4}{a} \tan^{-1}\left(\frac{x}{a}\right) + C}$

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(vi) $J = \int \frac{4x}{x^2 + a^2} \, dx$

Ans: $J = 2 \int \frac{2x}{x^2 + a^2} \, dx = \boxed{2 \ln[x^2 + a^2] + C}$

(vii) $K = \int \frac{4x^2}{x^2 + a^2} \, dx$

Ans: $K = 4 \int \frac{x^2 + a^2 - a^2}{x^2 + a^2} \, dx = 4\left[\int dx - a^2 \int \frac{dx}{x^2 + a^2}\right] = \boxed{4x - 4a \tan^{-1}\left(\frac{x}{a}\right) + C}$